CALCULATION OF NONSTATIONARY ELASTIC WAVES IN AN ISOTROPIC LAYER

V. A. Saraikin

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A plane problem of nonstationary waves in an infinite isotropic layer is considered. A normal force begins to act on the boundary of the layer at the instant t = 0. The opposite side of the layer is free from stresses. Using integral transformations, the solution of the problem is obtained in terms of transforms. Expanding the transform solution in a series of exponential powers and inverting each term of the resulting series, the exact solution of the problem is analytically determined. The fields of stresses and velocities in the layer are calculated. The use of analytical relationships for the calculation, in contrast to the calculation with finite-difference methods, allows us to fairly accurately determine the wave pattern and to eliminate the specific effects inherent in the difference equations. The calculation algorithm used in this work allows us to calculate the solutions of the problem at any point of the layer. The results presented give an idea about the distribution of stresses and velocities of particles across the thickness and in the longitudinal direction. The calculation of nonstationary problems by summing over waves, as is done in the present work, side by side with the methods presented in [1, 2], allows transient wave processes in the layer to be represented in a more complete manner.

Let a layer occupy a region bounded by the planes y = 0 and y = 1. The other two axes (x and z) are perpendicular to the y axis and are located in the plane of the layer. As units of measurement in the problem we have taken the thickness of the layer, the velocity of the wave of expansion (c_1), and the density of the material. The time interval during which the wave of expansion covers a distance equal to the thickness of the layer serves as a time unit.

In the boundary conditions we specify the distribution of the stress vector on the front side of the layer y = 0; on the rear side of the layer y = 1 stresses are absent:

$$\sigma_y = -\frac{1}{\pi}\frac{\delta}{\delta^2 + x^2}\delta_0(t), \quad \sigma_{xy} = 0 \quad (y = 0)$$
$$\sigma_y = \tau_{xy} = 0 \quad (y = 1) \tag{1}$$

Here by $\delta_0(t)$ we have denoted the Heaviside unit function, while δ is a real parameter of the load.

The initial conditions of the problem are zero.

The stresses and displacements do not depend on the z coordinate and are connected by the linear relationships

$$\sigma_x = \frac{\partial u}{\partial x} + (1 - 2c_2^2)\frac{\partial v}{\partial y}, \quad \sigma_{xy} = c_2^2\left(\frac{\partial u}{\partial y} + \frac{\partial}{\partial x}\right)$$
$$\sigma_y = (1 - 2c_2^2)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \tag{2}$$

where ε_{ij} are the strains in the layer.

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The longitudinal displacement u and the transverse displacement v in the layer must satisfy the equations

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + (1 - c_2^2)\frac{\partial^2 v}{\partial x\,\partial y} + c_2^2\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} = 0 \\
&c_2^2\frac{\partial^2 v}{\partial x^2} + (1 - c_2^2)\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial t^2} = 0
\end{aligned} \tag{3}
$$

In terms of the units of measurement adopted $c_2 = \sqrt{\mu}$ is the velocity of the shear wave; $\lambda + 2\mu = 1$; λ, μ are the elastic constants of the material, and t is the time.

When solving the problem, we use the Laplace transformation with respect to t and the Fourier transformation along the x axis

$$
u^L(p) = \int_0^\infty u(t)\exp(-pt)\,dt, \quad u^F(q) = \int_{-\infty}^{\infty} u(x)\exp(iqx)\,dx
$$

Having applied the integral transformations to the equations of motion (3), we obtain a system of ordinary differential equations with constant coefficients. Solving the system and determining the values of the constants in the general solution from the transformed boundary conditions (1), we find the transforms of stresses and velocities in the isotropic layer

$$
\begin{aligned}
&\left(\frac{\partial u}{\partial t}\right)^{LF} = \frac{1}{2} b_2^2 iq\exp(-\delta|q|)\left[(n_2^2 + q^2)\Omega_1 - 2n_1n_2\Omega_2\right] \\
&\left(\frac{\partial v}{\partial t}\right)^{LF} = \frac{1}{2} b_2^2 n_1\exp(-\delta|q|)\frac{d}{dy}\left[(n_2^2 + q^2)\frac{\Omega_1}{n_1} - 2q^2\frac{\Omega_2}{n_2}\right] \\
&\sigma_x^{LF} = \frac{1}{2p}\exp(-\delta|q|)\left[(n_2^2 + q^2)(b_2^2q^2 - (b_2^2 - 2)n_1^2)\Omega_1 - 4n_1n_2q^2\Omega_2\right] \\
&\sigma_y^{LF} = -\frac{1}{2p}\exp(-\delta|q|)\left[(n_2^2 + q^2)^2\Omega_1 - 4n_1n_2q^2\Omega_2\right] \\
&\sigma_{xy}^{LF} = \frac{iq}{p} n_1(n_2^2 + q^2)\exp(-\delta|q|)\frac{d}{dy}\left(\frac{\Omega_1}{n_1} - \frac{\Omega_2}{n_2}\right) \\
&n_1 = \sqrt{q^2 + p^2}, \quad n_2 = \sqrt{q^2 + b_2^2p^2}, \quad n_{1,2} > 0 \\
&\text{for} \quad p > 0, \quad \mathrm{Im}\,q = 0 \\
&\Omega_l = \frac{1}{L_1}\operatorname{sh}\frac{n_j}{2}\operatorname{ch}\left(\frac{n_l}{2} - n_ly\right) + \frac{1}{L_2}\operatorname{ch}\frac{n_j}{2}\operatorname{sh}\left(\frac{n_l}{2} - n_ly\right) \\
&(b_2 = 1/c_2, \quad l = 1, 2; \quad l + j = 3)
\end{aligned} \tag{4}
$$

By L_1 and L_2 we denote the denominators of the symmetric and antisymmetric parts of the solution

$$
L_l = (n_2^2 + q^2)^2\operatorname{ch}\frac{n_l}{2}\operatorname{sh}\frac{n_j}{2} - 4n_1n_2q^2\operatorname{sh}\frac{n_l}{2}\operatorname{ch}\frac{n_j}{2} \qquad (l = 1, 2; \quad l + j = 3)
$$

The solution (4) describes all waves that arise as a result of multiple reflections from the boundaries. We can represent the given solution in the form of a series of wave groups that undergo the same number of reflections. For this we represent the solution (4) in the form of expansions in a series of exponential powers (see, for example, [3])

$$
\begin{aligned}
&R_1L_l^{-1} = 4\exp\left(-\frac{n_1}{2} - \frac{n_2}{2}\right)\sum_{n=0}^{\infty}\left[\exp(-n_1 - n_2) + (-1)^l \times\right. \\
&\left.\times\,\gamma(\exp(-n_1) - \exp(-n_2))\right]^n = 4\exp\left(-\frac{n_1}{2} - \frac{n_2}{2}\right) \times \\
&\times \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}(-1)^{(2-l)k} a_{mnk}\exp\left[(m - n)n_1 + (k - m - n)n_2\right] \\
&\gamma = \frac{R_2}{R_1}, \quad R_l = (n_2^2 + q^2)^2 + (-1)^l 4n_1n_2q^2, \quad a_{mnk} = \frac{(-1)^m\gamma^k n!}{m!\,(n - k)!\,(k - m)!} \qquad (l = 1, 2)
\end{aligned}
$$

These expansions allow us to write Ω_l in the solution in the form of a sum of exponentials. Each exponential determines the contribution to the solution by a reflected wave corresponding to it

$$
\begin{aligned}
&R_1\Omega_l = \left[\exp(-n_ly) - \exp(-n_1 - n_2 + n_ly)\right]\Sigma_1 + \left[\exp(-n_l\right. \\
&\left. + n_ly) - \exp(-n_j - n_ly)\right]\Sigma_2
\end{aligned}
$$

$$\Sigma_l = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}[(-1)^k + (-1)^{l+1}]\, a_{mnk}\exp[(m-n)n_1 + (k-m-n)n_2] \quad (l = 1, 2;\ l + j = 3) \tag{5}$$

We substitute the expansions (5) into the solution of the problem (4). The exponential multipliers of each term in the solution indicate the delay of the original function for the given term. Thus, in the case of a given time t we must retain a finite number of terms in the solution. Below the summation of waves is written in detail.

We consider an arbitrary term in the solution (4)

$$\begin{gathered}\sigma^{LF}(p, q, y) = \frac{1}{p}\Phi(p, q)\exp[-\alpha n_1(p, q) - \beta n_2(p, q) - \\ - \delta|q|] = \sigma_0^{LF}(p, q)\exp(-\delta\,|\,q\,|) \\ n_{1,2}(p, p\xi) = p n_{1,2}(1, \xi), \quad \Phi(p, p\xi) = \Phi(1, \xi)\end{gathered} \tag{6}$$

Here $n_{1,2}$ are homogeneous functions of the first order, while Φ is of the zeroth order. Transforms of such form are inherent to simpler dynamic problems: the plane problem concerned with a disturbance source in an infinite medium, the plane problem of Lamb for a half-space. In the given case the solution of the problem with the characteristic dimensions (the thickness of the layer and the load parameter δ), thanks to the expansions (5), can be represented as a sum of similar LF transforms.

When inverting (6), we use the method proposed in [4] (pp. 80-85, 194-201) and used in the solution of the problem of Lamb for an isotropic half-space.

We represent the sought original function in (6) in the form

$$\sigma = \sigma_+(s_+) + \sigma_-(s_-), \qquad s_\pm = \delta \pm ix \tag{7}$$

The functions σ_+ and σ_- are the analytic functions $s_\pm$ in the right half-planes Re $s_\pm > 0$

$$\sigma_\pm(s_\pm) = \frac{1}{2\pi}\int_0^\infty \sigma_0^F(\pm q)\exp(-s_\pm q)\,dq \tag{8}$$

Since σ_+ and σ_- are analytic functions, they are completely determined by its values on the real half-axes Re $s_\pm = \delta > 0$. Taking into account the representation of the function σ in (7), from (8) we determine

$$\sigma^L = \sigma_+^L + \sigma_-^L = 2\,\mathrm{Re}\,\sigma_+^L = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \sigma_0^{LF}(p, q)\exp(-s_+ q)\,dq \tag{9}$$

Having put $q = p\xi$ in the case $s_+ > 0$, $p > 0$, we obtain the L transform of the sought function

$$\begin{gathered}\sigma^L(p, s_+, y) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \Phi(1, \xi)\exp[-p(\alpha m_1 + \beta m_2 + s_+\xi)]\,d\xi \\ m_1 = \sqrt{1 + \xi^2}, \qquad m_2 = \sqrt{b_2^2 + \xi^2} \\ m_{1,2} > 0 \quad \text{for} \quad \mathrm{Re}\,\xi > 0, \quad \mathrm{Im}\,\xi = 0\end{gathered} \tag{10}$$

The positive values of p lying on the right of the convergence abscissa completely determine the transform as an analytic function of p.

We denote

$$\alpha m_1(\xi) + \beta m_2(\xi) + s_+\xi = t + \alpha + b_2\beta \quad (s_+ > 0) \tag{11}$$

The positive solution of Eq. (11) is the only one relative to t, since on the left there is the positive ($s_+ > 0$) monotonically increasing function

$$(d\xi/dt)^{-1} = \alpha m'_1\ (\xi) + \beta m_2'\ (\xi) + s_+ > 0 \qquad (\xi > 0)$$
$$m_{1,2}\ (\xi) \to \infty \quad \text{as} \quad \xi \to \infty$$

Carrying out the substitution of the variables (11), we bring the L transform of the function σ in the expression (10) to the form

$$\sigma^L(p, s_+, y) = \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty \frac{\Phi(1, \xi(t, s_+, y))}{\alpha m_1' + \beta m_2' + s_+} \exp\left[-p(t + \alpha + b_2\beta)\right] dt \qquad (12)$$
$$(s_+ > 0)$$

On the right side (12) there stands the Laplace transform of the function

$$\sigma(t, s_+, y) = \frac{1}{\pi}\,\mathrm{Re}\left[\frac{\Phi(1, \xi(t - \alpha - b_2\beta, s_+, y))}{\alpha m_1' + \beta m_2' + s_+}\right] \qquad (13)$$
$$(s_+ = \delta + ix)$$

which thus is the sought function. In the case $\delta = 0$ we obtain the expression of the general term of the solution for a concentrated load.

When solving the problem on a digital computer, the expressions (13) were summed. The concrete form of $\Phi(1, \xi)$ for each term emerges from (4) and (5). The values of ξ were determined numerically from Eq. (11), which for $\alpha \neq 0$, $\beta \neq 0$, $s_+ = \delta + ix \neq 0$ is an equation of the fourth degree with complex coefficients. The degree of the equation is reduced if either α or β is zero. If $s_+ = 0$ (the load on the surface of the layer is concentrated and the wave process is considered on the axis of symmetry $x = 0$), then Eq. (11) is reduced to a biquadratic equation. Although in this case ξ is fairly simply expressed in terms of t and y, it is not regarded as possible to present the solution in explicit form because of its cumbersomeness.

The solution for the tangential action on the front surface of the layer is obtained analogously. If in the boundary conditions we replace $\delta_0(t)$ by the impulse $\delta_1(t)$, then also the u and v-displacements in the layer will be expressed.

We write the algorithm of wave summation used in the problem during the calculation. In the expansions of Ω_l, in the expressions (5), the quantities

$$A_1 = \exp(-n_1 - n_2 + n_1 y), \quad A_3 = \exp(-n_1 + n_1 y)$$
$$A_2 = \exp(-n_1 - n_2 + n_2 y), \quad A_4 = \exp(-n_2 + n_2 y)$$
$$B_1 = \exp(-n_1 y), \quad B_3 = \exp(-n_2 - n_1 y)$$
$$B_2 = \exp(-n_2 y), \quad B_4 = \exp(-n_1 - n_2 y)$$

stand as multipliers of Σ_1 and Σ_2.

The terms with the coefficients A_i describe waves going from the rear side of the layer $y = 1$ to the front side $y = 0$; the terms with the coefficients B_i describe waves reflected from the front side and going to the rear side of the layer. The radicals n_1 and n_2 of the indices of the exponentials indicate the form of the wave, while the multipliers in front of them [see (5)] indicate the length of run of the wave across the thickness of the layer.

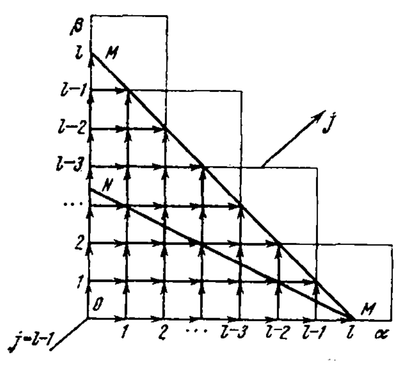

Fig. 1

We denote, as was done above [see (6)], by $\alpha(m, n)$ the multiplier in the index of the exponential for n_1, and denote by $\beta(m, n, k)$ the multiplier for n_2. Let j determine the number of reflections which the wave of expansion underwent, going out first from the disturbance source on the boundary $y = 0$. If t_0 is the time since the start of the process, then $0 \leq j \leq E(t_0) + 1$ (E is the integer part of the number). Collecting in Σ_1 and Σ_2 waves traversing the thickness of the layer j times ($\alpha + \beta = j$), we determine the combinations of m, n, k, which describe all waves that have been formed after the j-th reflection. In this way we single out all waves that have covered the same distance across the thickness of the layer. The fact that they proceed with different velocities (c_1 and c_2) and, consequently, that reflections of the shear waves are delayed in time, will be taken into account later.

We find that for Σ_1 and Σ_2 with the multipliers A_1, A_2, B_1, B_2 in the case of an even j, m, n, k, they vary within the limits $j/2 \le n \le j$, $k = 2n - j$, $0 \le m \le k$, while in the case of an odd j they vary within the limits $(j+1)/2 \le n \le j$, $k = 2n - j$, $0 \le m \le k$.

For waves described by the expression with the multipliers A_3, A_4, B_3, B_4 in the summation we must make a shift, and namely, if j is even, then for $j \ge 2$ we find $j/2 \le n \le j - 1$, $k = 2n - j + 1$, $0 \le m \le k$. If, however, j is odd and $j \ge 3$, then $(j-1)/2 \le n \le j - 1$, $k = 2n - j + 1$, $0 \le m \le k$.

In the calculation of the problem the number of reflections of waves from the surfaces of the layer was determined as follows. In Fig. 1 by arrows we have indicated the directions along which one must move to reach the straight line MM. Along the axes we have marked off the distance $OM = j + 1 = l$ traversed by the first wave of expansion up to the instant of $(j+1)$-th reflection. The nodes of the grid determine the values of α and β at which the reflections took place. All possible paths up to the straight line MM, equal to OM, determine all combinations of waves (the values α and β) which were formed after the j-th reflection. Having summed these waves (see the dependences of m, n, k on j presented above), we determine the contribution of the given j-th reflection to the solution of the problem. To find the complete solution of the problem, the sums obtained for the various j must be summed over j.

To take into account the delay of the shear waves, we mark off on the β axis the segment $ON = c_2 t_0$. The straight line MN in Fig. 1 (its equation is $\alpha + b_2\beta = t_0$) will cut off only those waves which managed to arrive at the given point of the layer, i.e., commencing with $j = E(c_2 t_0)$, we must take into account waves for which the inequality $t_p + t_s = \alpha + b_2\beta \le t_0$ is fulfilled. Here by t_p and t_s we have marked the times taken by the expansion and shear waves in traversing across the thickness of the layer over the distances α and β respectively.

In the calculation of the problem the values of ξ were determined from Eq. (11). For $s_+ > 0$ $(x = 0)$ positive ξ are the solutions of the equation. Extending the solution of the problem to values $x > 0$, we must analytically continue the solution of the equation into the domain of complex values. The values $x < 0$ can be ignored because of the symmetry of the problem about the y axis. From the analysis of the solutions of Eq. (11) we can determine the signs of the real and imaginary parts of ξ. It is shown that in the case $x > 0$ in the region where disturbances exist $\mathrm{Re}\,\xi \ge 0$, while $\mathrm{Im}\,\xi \le 0$. Thus, for positive values of x the roots of Eq. (11) are located in the fourth quadrant of the complex ξ plane.

When calculating the waves in the layer, we assumed $c_2 = 1/1.7$. This corresponds to $\lambda = 0.89\,\mu$. Below we have presented certain results of the calculations of the field of stresses and velocities of particles in the layer under a load that is close to a concentrated load ($\delta = 0.01$).

When a load distributed along the x axis acts at the instant of time $t = 0$, all points of the surface are subjected to a disturbance and it radiates expansion and shear waves. Straight-lined fronts propagating in the direction of the y axis with the velocity of expansion and shear waves are the envelopes of these waves. The finite jump undergone by the solution, when passing through the straight-lined front, decays with time. Within these regions the values of stresses and velocities of particles are continuous, but they have peaks at the points corresponding to the wave fronts in the case of a concentrated load. The peaks are expressed the more sharply, the closer the load is taken to a concentrated load. For a concentrated load the values of velocities of particles and stresses at these points become infinite. In the following, since the load is taken to be close to a concentrated load, precisely these points will be called frontal points.

On the surface $y = 0$ σ_y and σ_{xy} are subjected to the boundary conditions (1), while $\pi\sigma_x = -99.99$ in the case $x = 0$, $y = 0$, $t = 1.9$. Then, beginning with $x = 0.65$, the value of σ_x becomes positive and has maxima at the points of Rayleigh and shear waves

$$x = 1.9\, c_R t \sim 1.025, \quad \pi\sigma_x = 33.97; \quad x = 1.9\, c_2 \sim 1.11, \quad \pi\sigma_x = 0.75$$

Before the front of the shear wave, in the case of $x = 1.6$ σ_x again becomes compressive and has a minimum $\pi\sigma_x = -0.92$ at the point $x = 1.875$. Since disturbances reflected from the boundary $y = 1$ cannot reach the boundary $y = 0$, the wave field on the surface of the layer coincides with the wave field on the surface of the half-space.

On the lower graphs of Fig. 2 we have represented the stresses inside the layer for $y = 0.2$, $0 \le x \le 1.9$, $t = 1.9$, where $\pi\sigma_x$ is the dotted line, $\pi\sigma_y$ is the solid line, and $\pi\sigma_{xy}$ is the dash-dotted line. The fronts were

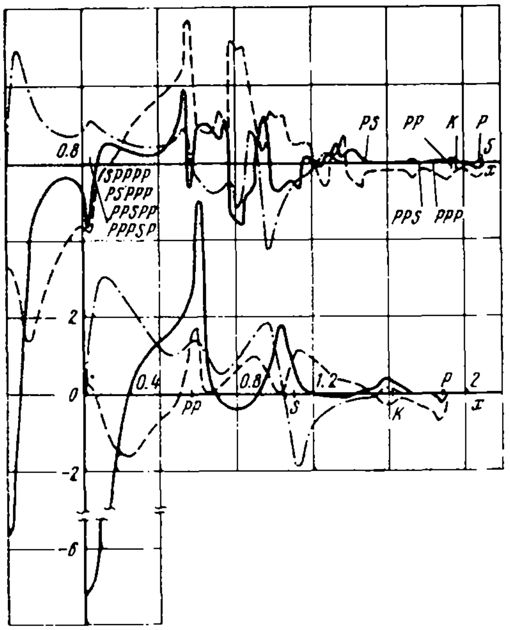

Fig. 2

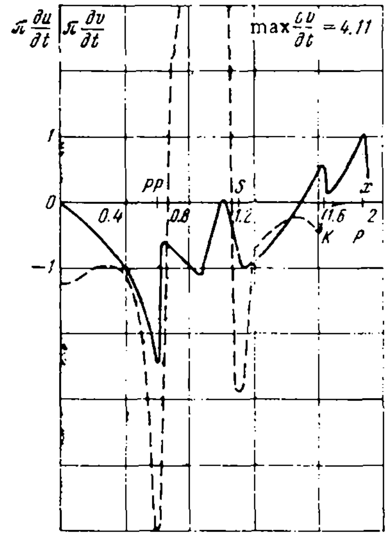

Fig. 3

constructed geometrically in order to determine the waves to which the peaks correspond. On the graph the symbols p and s denote the traces of the fronts of expansion and shear waves. The sequence of symbols points to the form of incident and reflected waves. The symbol k marks the trace of the straight-lined front of the shear wave. The presence of a peak of σ_x at $x = 0.9$ is due, apparently, to a Rayleigh surface wave. This peak, following directly the trace behind the peak on the front of the shear wave, is observed for $y = 0.4$, $x = 0.85$. The value of the stress σ_x at this point is

$$\pi\sigma_x = \pi (\sigma_x)_0 + \pi (\sigma_x)_{pp} = 0.181 + 0.589$$

Here we have isolated the contribution produced by the reflected longitudinal wave.

A calculation showed a growth in the disturbances at points of the straight-lined front k as y increases. The stress σ_x on the axis of symmetry $x = 0$, when moving away from the origin of the coordinates, from compressive becomes tensile and increases, while σ_y, decreasing to zero at $y = 1$, remains compressive all the time. The shear stress on the $x = 0$ axis is zero.

On the graphs of Fig. 2 we have also presented the values of the stresses $\pi\sigma_x$, $\pi\sigma_y$, $\pi\sigma_{xy}$ in the section $y = 0.2$, calculated for the instant of time $t = 5$. Since the solution was determined by summing over waves, the contribution of each reflected wave to the solution is easily determined. In each term we can single out a peak at the point of the front of the given wave. In this manner the traces of certain wave fronts shown on the graph were approximately determined. From the graphs thus presented we see that the stress σ_y most often changes its sign, oscillating about zero. The stress σ_x within a fairly large segment $0.7 < x < 2.4$ is compressive. The shear stress σ_{xy} changes the sign approximately as many times as there are reflected circular shear fronts. The shear stress changes the sign also when passing through the straight-lined front of the shear wave k.

The graphs in Fig. 3 represent the velocities of particles for $y = 0.2$ and $t = 1.9$ ($\partial u/\partial t$ is the solid line, $\partial v/dt$ is the dotted line). Using the example of the wave pp, we can examine the effect of the reflected wave on the direction of the result velocity of particles. If up to the arrival of the reflected wave at points within the s wave $\partial u/\partial t \leq 0$ and $\partial v/\partial t > 0$, then the reflected expansion wave pp, in which $\partial u/\partial t \leq 0$, $dv/dt < 0$, "swings about" the resulting velocity. Here $\partial u/\partial t \leq 0$, $\partial v/\partial t < 0$. The peaks at $x = 1.6$ determine the arrival of the straight-lined front of the shear wave. The resulting velocity is directed to the surface $y = 0$ and almost coincides in direction with the straight-lined front of the shear wave.

The calculations were carried out on a BÉSM-6 digital computer. When constructing the graphs, 77 points were taken on the segment $0 \leq x \leq 1.9$. For the calculation of σ_x, σ_y, σ_{xy}, $\partial u/\partial t$, $\partial v/\partial t$ at a single point for $t = 1.9$ approximately 0.16 sec was used.

We note the applicability of the calculation scheme presented in this paper to the determination of the solution of an axisymmetric problem. This follows from the possibility of going from the Fourier transformation over to the Hankel transformation [4, 5]. Formally in (13) we must replace x by $r \sin \theta$ (r is the radius from the origin of the coordinates) and integrate over the angle θ from 0 to π. The representation of the solution in the form of a series and its investigation on the axis of symmetry can be found in [6, 7].

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